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Functions $\sin_K x$ and $\cos_K x$

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Abstract. We introduce the piecewise linear functions $\sin_K x$ and $\cos_K x$ representing K straight line segments with vertices on $\sin x$ and $\cos x$ respectively. We Fourier expand these functions, discuss their properties, and derive a number of identities which follow from the expansion of the functions themselves and their integrals or derivatives. The motivation to study $\sin_K x$ and $\cos_K x$ comes from computer simulations of the Rayleigh–Taylor instability in which the eigenmodes $\sin x$ and $\cos x$ are represented by $\sin_K x$ and $\cos_K x$, $K+1$ being the number of nodes used in the simulations. We find that the harmonics generated by a finite K representation occur only at multiples of K plus or minus one.

1. Introduction

In this paper we introduce the functions $\sin_K x$ and $\cos_K x$ and discuss their properties. These are continuous piecewise linear functions representing $\sin x$ and $\cos x$ by K straight line segments, $K = 1, 2, 3, \ldots$. More specifically, we divide the interval $0 \leqslant x \leqslant 2\pi$ into K equal segments and in each segment, i.e., for $(2\pi/K)(j-1) \leqslant x \leqslant (2\pi/K) j$, $j = 1, 2, \ldots, K$, we define $\sin_K x$ as a straight line with vertices on $\sin(2\pi/K)(j-1)$ and $\sin(2\pi/K)j$. Similarly for $\cos_K x$. Examples are shown in figure 1.

The motivation to study these functions comes from computer simulations. The trigonometric functions arise in a large variety of physical problems and their simulation on a grid calls for resolving the functions by a finite number, say K, of straight line segments. Of course as $K \to \infty$ $\sin_K x \to \sin x$. But all computer simulations are carried out with finite K.

The specific application we have in mind is the simulation of the Rayleigh–Taylor instability [1, 2] where perturbations at the interface between two fluids grow under the action of accelerating forces like gravity (for a review see Sharp [3]). The eigenmodes, which are sinusoidal in shape, grow exponentially in time with a definite growth rate as long as the perturbation remains linear, i.e., the amplitude η is much smaller than the wavelength λ. We ran a number of test problems [4] to check the code and in particular to check how well it performed, i.e., reproduced the well known growth rate as K increases. Since the actual simulated shape was $\sin_K x$ and not $\sin x$, we were naturally interested in the properties of this function.

In addition, starting with a single wavelength we could track the generation of the higher harmonics which appear in the weakly nonlinear regime, $\eta \sim \lambda$, signalling deviations from a pure sinusoidal wave and the beginning of the bubble-and-spike regime [3, 5, 6]. Since $\sin_K x$ naturally has higher harmonics it was important to trace their source to make sure they are correctly generated by the algorithm of our hydrocode and that they were not simply a finite-K effect.

This is the motivation for the analytical work reported here. The functions $\sin_K x$ and $\cos_K x$ are Fourier expanded and their properties are discussed in the next section. We present concluding remarks in section 3, and mathematical details in an appendix.

2. Fourier expansion and properties of $\sin_K x$ and $\cos_K x$

2.1a. Fourier expansion

The evaluation of the Fourier coefficients is tedious and given in appendix A. The result can be written in an exceptionally compact form:

$$\sin_K x = \left(\frac{K}{\pi}\sin\frac{\pi}{K}\right)^2 \sum_{i=-\infty}^{\infty} \frac{\sin(iK+1)x}{(iK+1)^2} \tag{1a}$$

$$= \left(\frac{K}{\pi}\sin\frac{\pi}{K}\right)^2 \left\{\sin x + \sum_{i=1}^{\infty}\left[\frac{\sin(iK+1)x}{(iK+1)^2} - \frac{\sin(iK-1)x}{(iK-1)^2}\right]\right\} \tag{1b}$$

for $K \geqslant 2$. The first four functions are:

$$\sin_1 x = \sin_2 x = 0 \tag{2a}$$

$$\sin_3 x = \frac{27}{4\pi^2}\left\{\sin x - \frac{\sin 2x}{2^2} + \frac{\sin 4x}{4^2} - \frac{\sin 5x}{5^2} + \frac{\sin 7x}{7^2} + \ldots\right\} \tag{2b}$$

$$\sin_4 x = \frac{8}{\pi^2}\left\{\sin x - \frac{\sin 3}{3^2} + \frac{\sin 5x}{5^2} - \frac{\sin 7x}{7^2} + \frac{\sin 9x}{9^2} + \ldots\right\}. \tag{2c}$$

In figure 1(*a*) we plot $\sin_3 x$ and $\sin_4 x$.

We will refer to the $\sin x$ term in equation (1) as the fundamental; the other terms are the higher harmonics which are present because K is finite. The amplitude of the fundamental is $(K/\pi \sin \pi/K)^2$ which is less than 1. The harmonics all occur at multiples of K plus or minus 1. This had important implications for the physical problem we were interested in as we discuss later. Like $\sin x$, $\sin_K x$ is an odd function of x, i.e., $\sin_K(-x) = -\sin_K x$.

The derivation and the results for $\cos_K x$ are very similar:

$$\cos_K x = \left(\frac{K}{\pi}\sin\frac{\pi}{K}\right)^2 \sum_{i=-\infty}^{\infty} \frac{\cos(iK+1)x}{(iK+1)^2} \tag{3a}$$

$$= \left(\frac{K}{\pi}\sin\frac{\pi}{K}\right)^2 \left\{\cos x + \sum_{i=1}^{\infty}\left[\frac{\cos(iK+1)x}{(iK+1)^2} + \frac{\cos(iK-1)x}{(iK-1)^2}\right]\right\} \tag{3b}$$

for $K \geqslant 2$. The first four functions are:

$$\cos_1 x = 1 \tag{4a}$$

$$\cos_2 x = \cos_4 x = \frac{8}{\pi^2}\left\{\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \frac{\cos 7x}{7^2} + \frac{\cos 9x}{9^2} + \ldots\right\} \tag{4b}$$

$$\cos_3 x = \frac{27}{4\pi^2}\left\{\cos x + \frac{\cos 2x}{2^2} + \frac{\cos 4x}{4^2} + \frac{\cos 5x}{5^2} + \frac{\cos 7x}{7^2} + \ldots\right\}. \tag{4c}$$

In figure 1(*b*) we plot $\cos_2 x$ and $\cos_3 x$. Of course $\cos_2 x = \cos_4 x$ because both represent a "Vee", as seen in figure 1(*b*). These are the only degenerate cases—all the other $\cos_K x$ are distinct from each other.

$\mathrm{Cos}_K x$, like $\cos x$, is an even function of x: $\cos_K(-x) = \cos_K x$.

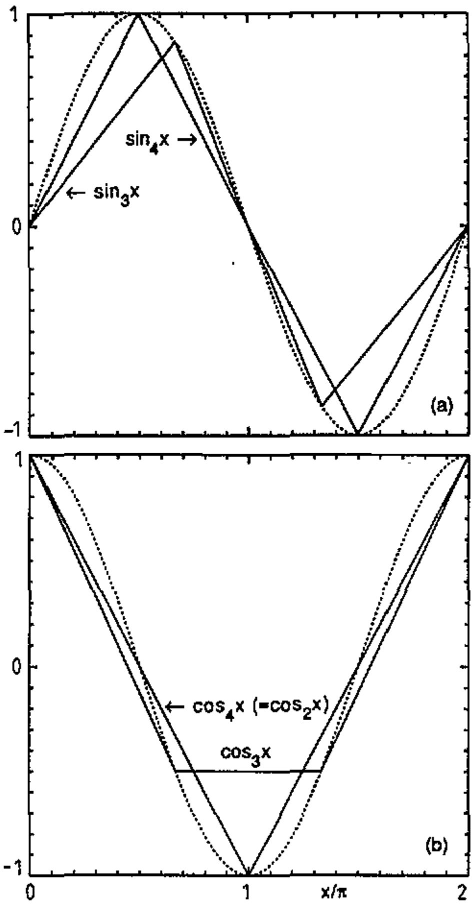

Figure 1. (*a*) The functions $\sin x$ (dotted), $\sin_3 x$ and $\sin_4 x$. The first two functions are $\sin_1 x = \sin_2 x = 0$. (*b*) The functions $\cos x$ (dotted), $\cos_3 x$ and $\cos_4 x$. The first two functions are $\cos_1 x = 1$ and $\cos_2 x = \cos_4 x$.

2.2b. Integrals and derivatives

Equations (1) and (3) can be repeatedly integrated because the Fourier expansions converge faster with each integration. Defining the operator I by

$$If(x) = \int_0^x f(x)\,\mathrm{d}x \tag{5}$$

we have

$$I \sin_K x = -\left(\frac{K}{\pi}\sin\frac{\pi}{K}\right)^2 \sum_{i=-\infty}^{\infty} \frac{\cos(iK+1)x - 1}{(iK+1)^3} \tag{6a}$$

$$= \frac{\pi}{K}\cot\frac{\pi}{K} - \left(\frac{K}{\pi}\sin\frac{\pi}{K}\right)^2 \sum_{i=-\infty}^{\infty} \frac{\cos(iK+1)x}{(iK+1)^3} \tag{6b}$$

where in the second step we have used the identity (see next subsection)

$$\sum_{i=-\infty}^{\infty} \frac{1}{(iK+1)^3} = \left(\frac{\pi}{K}\right)^3 \operatorname{cosec}^2\frac{\pi}{K}\cot\frac{\pi}{K}. \tag{7}$$

From equation (6) we obtain

$$I^2 \sin_K x = \frac{\pi}{K} x \cot \frac{\pi}{K} - \left(\frac{K}{\pi} \sin \frac{\pi}{K}\right)^2 \sum_{i=-\infty}^{\infty} \frac{\sin(iK+1)x}{(iK+1)^4} \tag{8}$$

and so forth.

Turning to $\cos_K x$ we have

$$I \cos_K x = \left(\frac{K}{\pi} \sin \frac{\pi}{K}\right)^2 \sum_{i=-\infty}^{\infty} \frac{\sin(iK+1)x}{(iK+1)^3} \tag{9}$$

and

$$I^2 \cos_K x = \left(\frac{K}{\pi} \sin \frac{\pi}{K}\right)^2 \sum_{i=-\infty}^{\infty} \frac{1-\cos(iK+1)x}{(iK+1)^4} \tag{10a}$$

$$= \frac{1}{3}\left(\frac{\pi}{K}\right)^2 \operatorname{cosec}^2 \frac{\pi}{K}\left(1 + 2\cos^2 \frac{\pi}{K}\right) - \left(\frac{K}{\pi} \sin \frac{\pi}{K}\right)^2 \sum_{i=-\infty}^{\infty} \frac{\cos(iK+1)x}{(iK+1)^4} \tag{10b}$$

where we have used the equality (see next subsection)

$$\sum_{i=-\infty}^{\infty} \frac{1}{(iK+1)^4} = \frac{1}{3}\left(\frac{\pi}{K}\right)^4 \operatorname{cosec}^4 \frac{\pi}{K}\left(1 + 2\cos^2 \frac{\pi}{K}\right). \tag{11}$$

The convergence is so fast that we get better than 10% accuracy by keeping only the fundamental, i.e., the $i = 0$ term in the above integrals.

We now turn to derivatives. Care must be exercised in differentiating equations (1) and (3) because the derivatives are not continuous. Defining the operator D by $Df(x) = \mathrm{d}f(x)/\mathrm{d}x$ we have

$$D \sin_K x = \left(\frac{K}{\pi} \sin \frac{\pi}{K}\right)^2 \sum_{i=-\infty}^{\infty} \frac{\cos(iK+1)x}{(iK+1)} \tag{12}$$

and

$$D \cos_K x = -\left(\frac{K}{\pi} \sin \frac{\pi}{K}\right)^2 \sum_{i=-\infty}^{\infty} \frac{\sin(iK+1)x}{(iK+1)}. \tag{13}$$

Needless to say, $D \sin_K x \neq \cos_K x$ just as $I \cos_K x \neq \sin_K x$, etc.

Since $\sin_K x$ and $\cos_K x$ are piecewise linear functions their derivatives are only piecewise continuous functions which are constant in each segment and jump to (generally) another constant value in adjacent segments, i.e., they have simple discontinuities at the vertices. In figure 2 we plot $D \sin_3 x$, $D \sin_4 x$, $D \cos_3 x$ and $D \cos_2 x (= D \cos_4 x)$.

The convergence in equations (12) and (13) is poorer than the convergence in any of the previous expansions. We illustrate with $D \sin_3 x$ and $D \cos_3 x$, which were already shown in figure 2, and see how well they are reproduced if we sum over $-10 \leqslant i \leqslant 10$ or over $-20 \leqslant i \leqslant 20$. The results are shown in figure 3. The oscillations around the discontinuous points are not damped by including more terms, leading to the well known Gibbs' phenomenon. Of course no such difficulties arise in the expansions of $\sin_K x$ and $\cos_K x$ or, better yet, their integrals.

Finally, we consider second derivatives. Since $D \sin_K x$ and $D \cos_K x$ are piecewise constant functions their derivatives $D^2 \sin_K x$ and $D^2 \cos_K x$ are identically zero everywhere except at the discontinuities, where they can be represented as Dirac delta

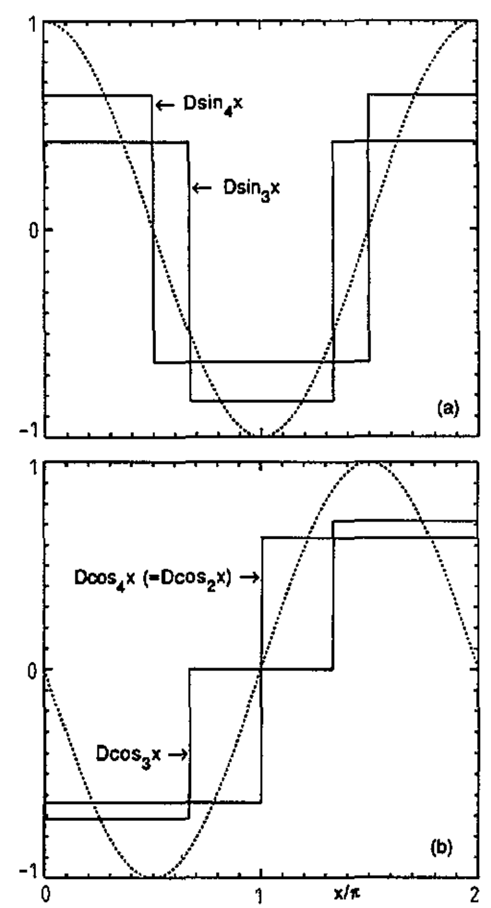

Figure 2. (a) The functions $\cos x$ (dotted), $D \sin_3 x$ and $D \sin_4 x$. (b) The functions $-\sin x$ (dotted), $D \cos_3 x$ and $D \cos_4 x$.

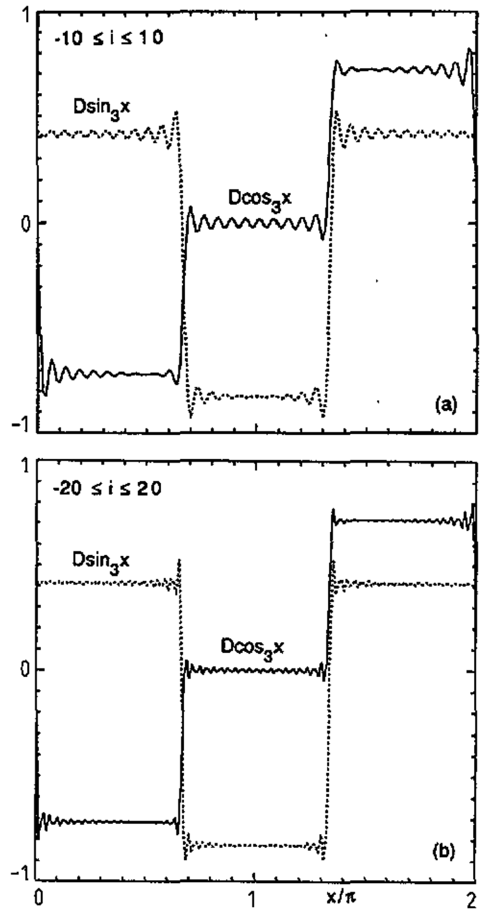

Figure 3. The functions $D \sin_3 x$ and $D \cos_3 x$ calculated with equations (12) and (13) keeping (a) twenty-one terms, $-10 \leqslant i \leqslant 10$, or ($b$) Forty-one terms, $-20 \leqslant i \leqslant 20$.

functions $\delta(x)$, multiplied by the appropriate jump $\Delta(D \sin_K x)$ or $\Delta(D \cos_K x)$. Here we have defined the jump in a function $f(x)$ by $\Delta f(x) = f(x_+) - f(x_-)$ where $x_\pm = x \pm \varepsilon$. Therefore,

$$D^2 \sin_K x = \sum_{j=0}^{K} \Delta \left(D \sin_K \frac{2\pi}{K} j \right) \delta \left(x - \frac{2\pi}{K} j \right) \tag{14}$$

and

$$D^2 \cos_K x = \sum_{j=0}^{K} \Delta \left(D \cos_K \frac{2\pi}{K} j \right) \delta \left(x - \frac{2\pi}{K} j \right). \tag{15}$$

The (constant) derivative in each segment is given by

$$D \sin_K x = \frac{K}{2\pi} \left(\sin \frac{2\pi}{K} j - \sin \frac{2\pi}{K} (j-1) \right) \tag{16}$$

and

$$D\cos_K x=\frac{K}{2\pi}\left(\cos\frac{2\pi}{K}j-\cos\frac{2\pi}{K}(j-1)\right)\qquad \frac{2\pi}{K}(j-1)<x<\frac{2\pi}{K}j. \tag{17}$$

Therefore the jump at each vertex is given by

$$\Delta\left(D\sin_K\frac{2\pi}{K}j\right)=\frac{K}{2\pi}\left(\sin\frac{2\pi}{K}(j+1)-\sin\frac{2\pi}{K}j-\sin\frac{2\pi}{K}j+\sin\frac{2\pi}{K}(j-1)\right)$$
$$=-2\frac{K}{\pi}\left(\sin\frac{\pi}{K}\right)^2\sin\frac{2\pi}{K}j \tag{18}$$

and, in a similar way,

$$\Delta\left(D\cos_K\frac{2\pi}{K}j\right)=-2\frac{K}{\pi}\left(\sin\frac{\pi}{K}\right)^2\cos\frac{2\pi}{K}j. \tag{19}$$

Equations (18) and (19) may be substituted into equations (14) and (15) respectively. With the understanding that $D^2\sin_K x$ and $D^2\cos_K x$ are sums over delta functions we can write

$$\frac{D^2\sin_K x}{D^2\cos_K x}=\tan x\qquad x=\frac{2\pi}{K}j. \tag{20}$$

This relationship can be obtained also by "formally" differentiating equations (1) and (3) twice (see appendix A).

2.3c. Identities

As mentioned in the Introduction we derived equations (1) and (3) to find out the "content" of $\sin_K x$ and $\cos_K x$, i.e., to find out how closely, for a given K, they resemble $\sin x$ and $\cos x$, what higher harmonics are present and what are their amplitudes. One may also view our results as simply a geometrical interpretation for the sums indicated on the right-hand side of those equations. Another application we take up in this subsection is the derivation of identities which follow from those equations and their derivatives or integrals.

We start by writing explicit expressions for $\sin_K x$ and $\cos_K x$:

$$\sin_K x=\frac{K}{2\pi}\left(\sin\frac{2\pi}{K}j-\sin\frac{2\pi}{K}(j-1)\right)x+(1-j)\sin\frac{2\pi}{K}j+j\sin\frac{2\pi}{K}(j-1) \tag{21}$$

and

$$\cos_K x=\frac{K}{2\pi}\left(\cos\frac{2\pi}{K}j-\cos\frac{2\pi}{K}(j-1)\right)x+(1-j)\cos\frac{2\pi}{K}j+j\cos\frac{2\pi}{K}(j-1) \tag{22}$$

for $(2\pi/K)(j-1)\leqslant x\leqslant(2\pi/K)j$, $j=1,2,3,\ldots,K$. One may view equations (21) and (22), taken with equations (1) and (3) respectively, as an infinite number of identities where x and K are the free variables. The geometrical interpretation of $\sin_K x$ and $\cos_K x$ is clearly more appealing than the explicit expressions given above.

Simpler identities, some of which are well known, can be derived by considering a specific x or K. For example, at $x=0$ $\cos_K 0=1$ and therefore

$$\sum_{i=-\infty}^{\infty}\frac{1}{(iK+1)^2}=\left(\frac{\pi}{K}\operatorname{cosec}\frac{\pi}{K}\right)^2 \tag{23}$$

which is a generalization of the familiar sum $1+1/3^2+1/5^2+1/7^2+\ldots=\pi^2/8$ obtained by setting $K=2$ or 4. An equally familiar sum is obtained from $\cos_2 x = 1-(2x/\pi)$ for $0 \leqslant x \leqslant \pi$ (see figure 1):

$$\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \frac{\cos 7x}{7^2} + \frac{\cos 9x}{9^2} + \ldots = \frac{\pi}{4}\left(\frac{\pi}{2} - x\right). \tag{24}$$

For $K=3$ we have

$$(\sin_3 x, \cos_3 x) = \left(\frac{3\sqrt{3}}{4\pi}x, 1 - \frac{9x}{4\pi}\right) \qquad 0 \leqslant x \leqslant \frac{2\pi}{3} \tag{25a}$$

$$= \left(\frac{3\sqrt{3}}{2}\left(1 - \frac{x}{\pi}\right), -\frac{1}{2}\right) \qquad \frac{2\pi}{3} \leqslant x \leqslant \frac{4\pi}{3} \tag{25b}$$

$$= \left(\frac{3\sqrt{3}}{2}\left(\frac{x}{2\pi} - 1\right), \frac{9x}{4\pi} - \frac{7}{2}\right) \qquad \frac{4\pi}{3} \leqslant x \leqslant 2\pi \tag{25c}$$

in each of the three segments $j=1, 2$ and 3. Therefore

$$\sin x - \frac{\sin 2x}{2^2} + \frac{\sin 4x}{4^2} - \frac{\sin 5x}{5^2} + \frac{\sin 7x}{7^2} + \ldots$$

$$= \frac{\pi}{3\sqrt{3}}x \qquad 0 \leqslant x \leqslant \frac{2\pi}{3} \tag{26a}$$

$$= \frac{2\pi}{3\sqrt{3}}(\pi - x) \qquad \frac{2\pi}{3} \leqslant x \leqslant \frac{4\pi}{3} \tag{26b}$$

$$= \frac{\pi}{3\sqrt{3}}(x - 2\pi) \qquad \frac{4\pi}{3} \leqslant x \leqslant 2\pi. \tag{26c}$$

From $\cos_3 x$ we have

$$\cos x + \frac{\cos 2x}{2^2} + \frac{\cos 4x}{4^2} + \frac{\cos 5x}{5^2} + \frac{\cos 7x}{7^2} + \ldots$$

$$= \frac{\pi}{3}\left(\frac{4\pi}{9} - x\right) \qquad 0 \leqslant x \leqslant \frac{2\pi}{3} \tag{27a}$$

$$= -\frac{2\pi^2}{27} \qquad \frac{2\pi}{3} \leqslant x \leqslant \frac{4\pi}{3} \tag{27b}$$

$$= \frac{\pi}{3}\left(x - \frac{14\pi}{9}\right) \qquad \frac{4\pi}{3} \leqslant x \leqslant 2\pi. \tag{27c}$$

From $\sin_4 x = 2x/\pi$ in the first segment (see figure 1) we get

$$\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \frac{\sin 7x}{7^2} + \frac{\sin 9x}{9^2} + \ldots = \frac{\pi x}{4} \tag{28}$$

for $-\pi/2 \leqslant x \leqslant \pi/2$, etc. We have used $\sin_K(-x) = -\sin_K x$.

We now turn briefly to identities derived from the derivatives:

$$D \sin_K x = \frac{K}{2\pi}\left(\sin\frac{2\pi}{K}j - \sin\frac{2\pi}{K}(j-1)\right) = \left(\frac{K}{\pi}\sin\frac{\pi}{K}\right)^2 \sum_{i=-\infty}^{\infty} \frac{\cos(iK+1)x}{iK+1} \tag{29}$$

and

$$D\cos_K x=\frac{K}{2\pi}\left(\cos\frac{2\pi}{K}j-\cos\frac{2\pi}{K}(j-1)\right)=-\left(\frac{K}{\pi}\sin\frac{\pi}{K}\right)^2\sum_{i=-\infty}^{\infty}\frac{\sin(iK+1)x}{iK+1}. \tag{30}$$

An important point to keep in mind is that these equations do *not*, in general, hold at the vertices but only *within* each segment, i.e., $(2\pi/K)(j-1)<x<(2\pi/K)j$, as we discussed in the previous subsection. We say "in general" because there may be vertices where the derivatives $D\sin_K x_\pm$ and $D\cos_K x_\pm$ happen to be continuous and then one is permitted to use the above equations at those exceptional vertices also. The best example is of course $\cos_4 x$ which is continuous at $x=\pi/2$ and at $x=3\pi/2$ because $\cos_4 x=\cos_2 x$. Another exceptional vertex is $x=0$ where $D\sin_K x$ (but *not* $D\cos_K x$) is continuous.

For illustration let us consider $D\sin_K x$ in the first segment where $D\sin_K x=K/2\pi\sin 2\pi/K$. Applying equation (29) at $x=0$ we get

$$D\sin_K 0=\frac{K}{2\pi}\sin\frac{2\pi}{K}=\left(\frac{K}{\pi}\sin\frac{\pi}{K}\right)^2\sum_{i=-\infty}^{\infty}\frac{1}{iK+1} \tag{31}$$

which gives the identity

$$\sum_{i=-\infty}^{\infty}\frac{1}{iK+1}=\frac{\pi}{K}\cot\frac{\pi}{K}. \tag{32}$$

At $x=\pi/K$, i.e., in the middle of the first segment, we have

$$D\sin_K\frac{\pi}{K}=\frac{K}{2\pi}\sin\frac{2\pi}{K}=\left(\frac{K}{\pi}\sin\frac{\pi}{K}\right)^2\sum_{i=-\infty}^{\infty}\frac{\cos(iK+1)(\pi/K)}{iK+1} \tag{33}$$

which gives the identity

$$\sum_{i=-\infty}^{\infty}\frac{(-1)^i}{iK+1}=\frac{\pi}{K}\operatorname{cosec}\frac{\pi}{K}. \tag{34}$$

Clearly, we cannot apply equation (29) at $x=2\pi/K$, the end of the first segment (we get a wrong identity). Similarly, we can apply equation (30) anywhere *within* the first segment such as the middle (we get the same equation (34) as above) but we *cannot* apply it at $x=0$ or, for that matter, at $x=2\pi/K$ (except for $K=4$).

Identities obtained from the derivatives are perhaps more interesting because they appear like "sum rules", i.e., the left-hand side of equations (29) and (30) are constant within each segment. For example, equation (34) is a special case of

$$\sum_{i=-\infty}^{\infty}\frac{\sin(iK+1)x}{iK+1}=\frac{\pi}{K}\qquad 0<x<\frac{2\pi}{K}. \tag{35}$$

As an example of a specific K let us consider $K=3$ again. The derivatives are

$$D\sin_3 x=3\sqrt{3}/4\pi, -3\sqrt{3}/2\pi, \text{ and } 3\sqrt{3}/4\pi \tag{36}$$

$$D\cos_3 x=-9/4\pi, 0, \text{ and } 9/4\pi \tag{37}$$

in the three segments $j = 1, 2$ and 3 respectively (see figure 2). Differentiating equations (26) and (27) we get

$$\cos x - \frac{\cos 2x}{2} + \frac{\cos 4x}{4} - \frac{\cos 5x}{5} + \frac{\cos 7x}{7} + \ldots$$

$$= \frac{\pi}{3\sqrt{3}} \qquad 0 \leqslant x \leqslant \frac{2\pi}{3} \tag{38a}$$

$$= -\frac{2\pi}{3\sqrt{3}} \qquad \frac{2\pi}{3} < x < \frac{4\pi}{3} \tag{38b}$$

$$= \frac{\pi}{3\sqrt{3}} \qquad \frac{4\pi}{3} < x \leqslant 2\pi \tag{38c}$$

and

$$\sin x + \frac{\sin 2x}{2} + \frac{\sin 4x}{4} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \ldots$$

$$= \frac{\pi}{3} \qquad 0 < x < \frac{2\pi}{3} \tag{39a}$$

$$= 0 \qquad \frac{2\pi}{3} < x < \frac{4\pi}{3} \tag{39b}$$

$$= -\frac{\pi}{3} \qquad \frac{4\pi}{3} < x < 2\pi. \tag{39c}$$

Let us now illustrate equation (20) with $K = 3$. As we discussed earlier, this equation relates the jumps in the derivatives at the vertices where the derivatives are, in general, discontinuous. It is trivially satisfied for *any* K at $x = 0$, π and 2π because $D \sin_K x$ is continuous at those points. This leaves two vertices for $K = 3$ where the derivatives are discontinuous. From equation (36) we have $D \sin_3 x = 3\sqrt{3}/4\pi$ and $-3\sqrt{3}/2\pi$ at $x = 2\pi/3 - \varepsilon$ and $2\pi/3 + \varepsilon$ respectively, and therefore $\Delta(D \sin_3(2\pi/3)) = -9\sqrt{3}/4\pi$. From equation (37) we have $D \cos_3 x = -9/4\pi$ and 0 at $x = 2\pi/3 - \varepsilon$ and $2\pi/3 + \varepsilon$ respectively, hence $\Delta(D \cos_3(2\pi/3)) = 9/4\pi$. The ratio of the discontinuities is $-9\sqrt{3}/4\pi \div 9/4\pi = -\sqrt{3} = \tan(2\pi/3)$. Similarly at the other vertex where $x = 4\pi/3$ and $\Delta(D \sin_3(4\pi/3)) = 9\sqrt{3}/4\pi$ while $\Delta(D \cos_3(4\pi/3)) = 9/4\pi$: their ratio, $\sqrt{3}$, equals $\tan(4\pi/3)$.

Finally, we consider identities derived from the integrals $I \sin_K x$, etc. Clearly, the first and higher integrals will produce sum rules involving x^2 and higher powers of x. We will not go into any detailed discussion except point out that all of the vertices are now legitimately included in the resulting highly convergent series. The identity given in equation (7), for example, can be derived by evaluating equation (6a) at $x = 2\pi/K$: since $\sin_K x = (Kx/2\pi)\sin(2\pi/K)$ in the first segment, $I \sin_K(2\pi/K) = (\pi/K)\sin(2\pi/K)$. Using $\cos(iK+1)x - 1 = \cos(2\pi i + (2\pi/K)) - 1 = \cos(2\pi/K) - 1 = -2\sin^2(\pi/K)$ in the right-hand side of equation (6a) we arrive at (7). Similarly for equation (11). Finally, we quote the following identities,

$$\sum_{i=-\infty}^{\infty} \frac{(-1)^i}{(iK+1)^5} = \left(\frac{\pi}{K}\operatorname{cosec}\frac{\pi}{K}\right)^5 \left\{1 - \frac{5}{6}\sin^2\frac{\pi}{K} + \frac{1}{24}\sin^4\frac{\pi}{K}\right\} \tag{40a}$$

and

$$\sum_{i=-\infty}^{\infty} \frac{1}{(iK+1)^5} = \left(\frac{\pi}{K} \operatorname{cosec} \frac{\pi}{K}\right)^5 \cos \frac{\pi}{K} \left(1 - \frac{1}{3} \sin^2 \frac{\pi}{K}\right) \tag{40b}$$

which we obtained by evaluating $I^3 \cos_K x$ in the middle and at the end of the first segment, i.e., at $x = \pi/K$ and $2\pi/K$ respectively.

2.4d. Orthogonality and normalization

We do not expect $\sin_K x$ and $\sin_{K'} x$ to be orthogonal to each other, and indeed they are not. The functions that *may* be orthogonal are $\sin_K Lx$ and $\sin_K L'x$ where

$$\sin_K Lx = \left(\frac{K}{\pi} \sin \frac{\pi}{K}\right)^2 \sum_{i=-\infty}^{\infty} \frac{\sin(iK+1)Lx}{(iK+1)^2} \tag{41}$$

and, similarly,

$$\cos_K Lx = \left(\frac{K}{\pi} \sin \frac{\pi}{K}\right)^2 \sum_{i=-\infty}^{\infty} \frac{\cos(iK+1)Lx}{(iK+1)^2}. \tag{42}$$

The geometrical interpretation of these functions is straightforward: within the interval $0 \leqslant x \leqslant 2\pi$ there are L complete sine and cosine cycles, each cycle being represented by K straight line segments. In analogy with $\sin Lx$ we might expect that $\sin_K Lx$ and $\sin_K L'x$ are orthogonal, and similarly for $\cos_K Lx$ and $\cos_K L'x$.

Unfortunately only $\sin_K Lx$ and $\cos_K L'x$ are orthogonal, i.e.,

$$\frac{1}{\pi} \int_0^{2\pi} \sin_K Lx \cos_K L'x \, dx = 0 \tag{43}$$

for any K, L and L'. Our original hope, again in analogy with the ordinary sine and cosine functions was that

$$\frac{1}{\pi} \int_0^{2\pi} \sin_K Lx \sin_K L'x \, dx \sim \delta_{L,L'} \tag{44a}$$

and

$$\frac{1}{\pi} \int_0^{2\pi} \cos_K Lx \cos_K L'x \, dx \sim \delta_{L,L'}. \tag{44b}$$

While these relations are certainly correct for $K \to \infty$ and hold for several values of K, L and L', they are *not* correct for arbitrary K. A counter example will suffice and we quote,

$$\frac{1}{\pi} \int_0^{2\pi} \cos_3 x \cos_3 2x \, dx = 1/8 \tag{45}$$

instead of zero. The functions $\cos_3 x$, $\cos_3 2x$ and their product are shown in figure 4(a).

Instead of equation (44), however, we found a rather surprising orthogonality relation which has no counterpart with the ordinary sine and cosine functions:

$$\frac{1}{\pi} \int_0^{2\pi} \sin_K Lx \sin_L Kx \, dx = \frac{1 + 2\cos^2(\pi/K)}{3} (1 - \delta_{K,1})(1 - \delta_{K,2}) \delta_{K,L} \tag{46a}$$

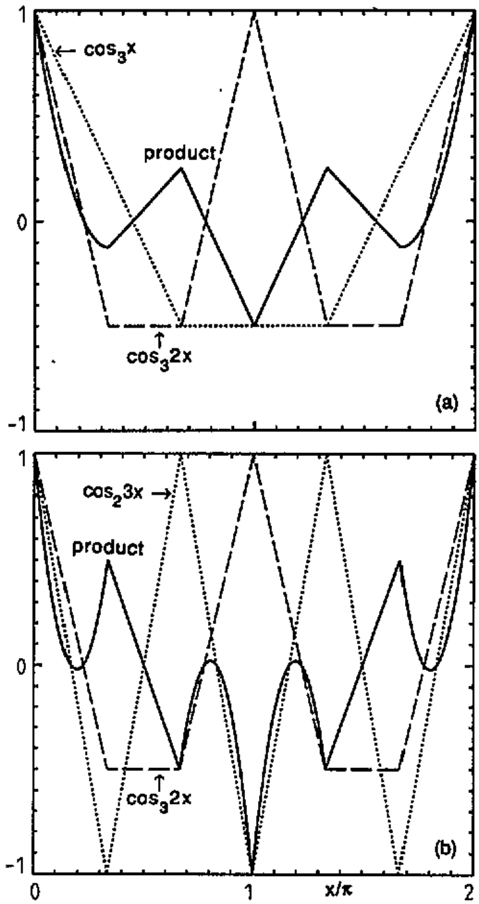

Figure 4. (*a*) The functions $\cos_3 x$ (dotted), $\cos_3 2x$ (dashed) and their product (continuous). The integral over the product is $\pi/8$—see equation (45). (*b*) The functions $\cos_2 3x$(dotted), $\cos_3 2x$ (dashed) and their product (continuous). The integral over the product is zero—see equation (47).

and

$$\frac{1}{\pi}\int_0^{2\pi} \cos_K Lx \cos_L Kx \, dx = \frac{1+2\cos^2(\pi/K)}{3}(1+\delta_{K,1})(1+\delta_{K,2})\delta_{K,L}. \tag{46b}$$

In other words $\sin_K Lx$ and $\sin_L Kx$ are orthogonal to each other and similarly for $\cos_K Lx$ and $\cos_L Kx$.

The surprising aspect of equation (46) comes from the fact that K and L play quite different roles (see equations (41) and (42))—$\sin_K Lx$ is L cycles resolved by K lines while $\sin_L Kx$ is K cycles resolved by L lines. For example

$$\frac{1}{\pi}\int_0^{2\pi} \cos_2 3x \cos_3 2x \, dx = 0 \tag{47}$$

which must be compared with equation (45). The functions $\cos_2 3x$, $\cos_3 2x$ and their product are plotted in figure 4(*b*).

The derivation of equation (46) is given in the Appendix. It gives us no physical insight as to why $\sin_K Lx$ and $\sin_L Kx$ are orthogonal.

Finally we discuss the normalization of $\sin_K Lx$ and $\cos_K Lx$. We find

$$\frac{1}{\pi}\int_0^{2\pi} (\sin_K Lx)^2\,\mathrm{d}x = \frac{1+2\cos^2(\pi/K)}{3}(1-\delta_{K,1})(1-\delta_{K,2}) \tag{48a}$$

and

$$\frac{1}{\pi}\int_0^{2\pi} (\cos_K Lx)^2\,\mathrm{d}x = \frac{1+2\cos^2(\pi/K)}{3}(1+\delta_{K,1})(1+\delta_{K,2}) \tag{48b}$$

for any L. For $L=K$ they agree with equation (46).

3. Concluding remarks

We have studied most of the mathematical properties of the functions $\sin_K x$ and $\cos_K x$. Our basic results are the Fourier expansions given in equations (1) and (3) respectively, from which the integrals $I\sin_K x$, $I\cos_K x$ and the derivatives $D\sin_K x$, $D\cos_K x$ follow immediately. The second derivatives must be taken with caution. From these expansions we obtain a plethora of identities some of which are well known. However, the point we wish to make here is that they were extremely easy to derive and the geometrical pictures associated with $\sin_K x$ and $\cos_K x$ make their regions of validity perfectly clear.

Though we have considered only positive integers K, the expansions (1) and (3) can be taken to define $\sin_K x$ and $\cos_K x$ for arbitrary K, except at the zeros of $iK \pm 1$, $i=1,2,\ldots$. We will not pursue this issue except to point out that $\sin_K x$ and $\cos_K x$ are both even functions of K, i.e., $\sin_{-K} x = \sin_K x$ and $\cos_{-K} x = \cos_K x$.

We are puzzled by the orthogonality of $\sin_K Lx$ and $\sin_L Kx$, but we are also disappointed that $\sin_K Lx$ and $\sin_K L'x$ are not, in general, orthogonal to each other.

Returning to the Rayleigh–Taylor instability, the physical problem which motivated our study, we note that equations (1) and (3) yield the following highly useful information: the harmonics generated by representing $\sin x$ by $\sin_K x$ and $\cos x$ by $\cos_K x$ all occur at *multiples of K plus or minus 1*. This has the following important consequence: these harmonics of wavelength $\lambda/(K\pm1)$, $\lambda/(2K\pm1)$, etc. ($\lambda = 2\pi$ here) are far from the physical harmonics $\lambda/2$, $\lambda/3$, etc., generated by the evolution of the Rayleigh–Taylor instability into the weakly nonlinear regime. In other words the physical evolution can be described as $\eta\cos x \to \eta\cos x + \eta^2\cos 2x + \eta^3\cos 3x + \ldots$, while the discretization with a typical value like $K=20$ gives $\eta\,(\cos x + \cos 19x/(19)^2 + \cos 21x/(21)^2 + \cos 39x/(39)^2 + \ldots)$. In short, the harmonics do not overlap. Therefore any second or third harmonic generated in a computer simulation must come from physical nonlinearity and, if the differencing algorithm of the code is accurate, the growth rate of the fundamental and at least the first harmonic must be correctly reproduced. At present hydrocodes are tested on the growth rate of the fundamental only [4, 7].

A completely different application is the evaluation of Fourier transforms,

$$F(k) = \int F(x)\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x \tag{49}$$

for arbitrary functions $F(x)$. One well known approach, among many, to evaluate the above integral is to use a generalized Gauss–Laguerre quadrature of order N [9]. We may think of this approach as approximating the integrand by a polynomial of order N. The alternative approach we are suggesting is to make the replacement $e^{ikx} \to \cos_K kx + i \sin_K kx$. In this approach the trigonometric functions are approximated while $F(x)$ is left intact. The advantage is that within each interval one needs to evaluate only $\int F(x)\,dx$ and $\int xF(x)\,dx$ which are simpler than $\int \sin kx F(x)\,dx$ or $\int \cos kx F(x)\,dx$. The efficiency and the convergence with K of our proposed method remains to be studied.

Clearly, in this paper we have solved the simplest problem of this type, namely, representing the ordinary sine and cosine functions by piecewise linear functions. Extensions readily come to mind: the same approach can be applied to other basis functions; alternatively, instead of using a linear function in each segment one may use a quadratic function or, more generally, a polynomial of degree m. In this view the generalized functions are, say, $\sin_K{}^m x$ and $\cos_K{}^m x$ which approach $\sin x$ and $\cos x$ as $K \to \infty$ or, independently, as $m \to \infty$. We have considered only $\sin_K{}^1 x$ and $\cos_K{}^1 x$.

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Appendix A

To derive equation (1) we Fourier expand $\sin_K x$ in the interval $0 \leq |x| \leq 2\pi$. Using the notation of Whittaker and Watson [8] we have

$$\sin_K x = \sum_{n=1}^{\infty} b_n(K) \sin \frac{nx}{2} \tag{A.1}$$

with the coefficients given by

$$b_n(K) = \frac{1}{\pi} \int_0^{2\pi} \sin_K x \sin \frac{nx}{2}\,dx$$

$$= \frac{1}{\pi} \sum_{j=1}^{K} \int_{\alpha(j-1)}^{\alpha j} \sin_K x \sin \frac{nx}{2}\,dx \tag{A.2}$$

where $\alpha \equiv 2\pi/K$. We have broken up the integration region into K equal segments.

Since in each segment $\sin_K x$ is a linear function of x we can use

$$\int f(x) \sin \frac{nx}{2}\,dx = \frac{2}{n}\left[-f(x) \cos \frac{nx}{2} + \frac{2}{n} f' \sin \frac{nx}{2}\right] \tag{A.3}$$

valid for any linear function, i.e., $f' = df/dx = \text{constant}$. We will see below that when we sum over j only the derivative terms will survive.

At the upper and lower limits $\sin_K x = \sin \alpha j$ and $\sin \alpha(j-1)$ respectively, and its derivative is of course $(\sin \alpha j - \sin \alpha(j-1))/\alpha$. Therefore

$$b_n(K) = \frac{2}{n\pi}\sum_{j=1}^{K}\left\{-\sin \alpha j \cos\frac{n\alpha j}{2} + \sin \alpha(j-1)\cos\frac{n\alpha(j-1)}{2} + \frac{2}{n\alpha}(\sin \alpha j - \sin \alpha(j-1))\left(\sin\frac{n\alpha j}{2} - \sin\frac{n\alpha(j-1)}{2}\right)\right\}. \tag{A.4}$$

Using trigonometric identities we expand equation (A.4),

$$\begin{aligned} b_n(K) = \frac{1}{n\pi}\sum_{j=1}^{K}\Bigg\{&\left[-1+\cos\alpha_+ + \frac{2}{n\alpha}\left(\sin\frac{n\alpha}{2} - \sin\alpha - \sin\alpha_+\right)\right]\sin j\alpha_+ \\ &+\left[1-\cos\alpha_- + \frac{2}{n\alpha}\left(-\sin\frac{n\alpha}{2} + \sin\alpha + \sin\alpha_-\right)\right]\sin j\alpha_- \\ &+\left[-\sin\alpha_+ + \frac{2}{n\alpha}\left(-1+\cos\frac{n\alpha}{2} + \cos\alpha - \cos\alpha_+\right)\right]\cos j\alpha_+ \\ &+\left[\sin\alpha_- + \frac{2}{n\alpha}\left(1-\cos\frac{n\alpha}{2} - \cos\alpha + \cos\alpha_-\right)\right]\cos j\alpha_-\Bigg\} \end{aligned} \tag{A.5}$$

where $\alpha_\pm \equiv \alpha(n/2 \pm 1) = 2\pi/K(n/2 \pm 1)$. The derivative terms which will survive summing over j can be recognized as the bracketed terms with the coefficient $2/n\alpha$. They can be written in the following compact forms

$$\begin{aligned} \sin\frac{n\alpha}{2} - \sin\alpha - \sin\alpha_+ &= 4\sin\frac{n\alpha}{4}\sin\frac{\alpha}{2}\sin\frac{\alpha_+}{2} \\ -\sin\frac{n\alpha}{2} + \sin\alpha + \sin\alpha_- &= 4\sin\frac{n\alpha}{4}\sin\frac{\alpha}{2}\sin\frac{\alpha_-}{2} \\ -1+\cos\frac{n\alpha}{2} + \cos\alpha - \cos\alpha_+ &= 4\sin\frac{n\alpha}{4}\sin\frac{\alpha}{2}\cos\frac{\alpha_+}{2} \\ 1-\cos\frac{n\alpha}{2} - \cos\alpha + \cos\alpha_- &= 4\sin\frac{n\alpha}{4}\sin\frac{\alpha}{2}\cos\frac{\alpha_-}{2}. \end{aligned} \tag{A.6}$$

We can now write equation (A.5) as

$$\begin{aligned} b_n(K) &= \frac{2}{n\pi}\sum_{j=1}^{K}\Bigg\{\sin\frac{\alpha_+}{2}\left[-\sin\frac{\alpha_+}{2} + \frac{4}{n\alpha}\sin\frac{n\alpha}{4}\sin\frac{\alpha}{2}\right]\sin j\alpha_+ \\ &\quad+\sin\frac{\alpha_-}{2}\left[\sin\frac{\alpha_-}{2} + \frac{4}{n\alpha}\sin\frac{n\alpha}{4}\sin\frac{\alpha}{2}\right]\sin j\alpha_- \\ &\quad+\cos\frac{\alpha_+}{2}\left[-\sin\frac{\alpha_+}{2} + \frac{4}{n\alpha}\sin\frac{n\alpha}{4}\sin\frac{\alpha}{2}\right]\cos j\alpha_+ \\ &\quad+\cos\frac{\alpha_-}{2}\left[\sin\frac{\alpha_-}{2} + \frac{4}{n\alpha}\sin\frac{n\alpha}{4}\sin\frac{\alpha}{2}\right]\cos j\alpha_-\Bigg\} \\ &= \frac{2}{n\pi}\sum_{j=1}^{K}\Bigg\{\left[-\sin\frac{\alpha_+}{2} + \frac{4}{n\alpha}\sin\frac{n\alpha}{4}\sin\frac{\alpha}{2}\right]\cos\left(j\alpha_+ - \frac{\alpha_+}{2}\right) \\ &\quad+\left[\sin\frac{\alpha_-}{2} + \frac{4}{n\alpha}\sin\frac{n\alpha}{4}\sin\frac{\alpha}{2}\right]\cos\left(j\alpha_- - \frac{\alpha_-}{2}\right)\Bigg\}. \end{aligned} \tag{A.7}$$

The sum over j can be carried out using well known identities, with the result

$$b_n(K)=\frac{1}{n\pi}\left\{\left[-\sin\frac{\alpha_+}{2}+\frac{4}{n\alpha}\sin\frac{n\alpha}{4}\sin\frac{\alpha}{2}\right]\frac{\sin K\alpha_+}{\sin(\alpha_+/2)}\right.$$
$$\left.+\left[\sin\frac{\alpha_-}{2}+\frac{4}{n\alpha}\sin\frac{n\alpha}{4}\sin\frac{\alpha}{2}\right]\frac{\sin K\alpha_-}{\sin(\alpha_-/2)}\right\}$$
$$=\frac{1}{n\pi}\left\{\sin K\alpha_- - \sin K\alpha_+\right.$$
$$\left.+\frac{4}{n\alpha}\sin\frac{n\alpha}{4}\sin\frac{\alpha}{2}\left[\frac{\sin K\alpha_+}{\sin(\alpha_+/2)}+\frac{\sin K\alpha_-}{\sin(\alpha_-/2)}\right]\right\}. \tag{A.8}$$

Now, $K\alpha_\pm=K(2\pi/K)(n/2\pm 1)=n\pi\pm 2\pi$, therefore $\sin K\alpha_\pm=0$ and the first two terms in equation (A.8) do not contribute, confirming our earlier statement that only the derivative terms survive the sum over j. By the same token these derivative terms also vanish *except when their denominators* $\sin(\alpha_\pm/2)$ *vanish*, which happens when $\alpha_\pm/2=i\pi$, $i=0,1,2$, etc., i.e., when $n/2=iK-1$ ($\sin(\alpha_+/2)=0$) or when $n/2=iK+1$ ($\sin(\alpha_-/2)=0$).

To find the ratios indicated in equation (A.8) we use the following expansion

$$\sin K\alpha_\pm = 2K\cos\frac{\alpha_\pm}{2}\sin\frac{\alpha_\pm}{2}+\text{higher terms in }\sin\frac{\alpha_\pm}{2}$$

and therefore the ratios appearing in equation (A.8) can be written as

$$\frac{\sin K\alpha_\pm}{\sin(\alpha_\pm)/2}=\left(2K\cos\frac{\alpha_\pm}{2}\right)\delta_{n/2,iK\mp 1}$$
$$=(2K\cos i\pi)\delta_{n/2,iK\mp 1}$$
$$=2K(-1)^i\delta_{n/2,iK\mp 1}. \tag{A.9}$$

Dropping the first two terms in equation (A.8) and using the above equation in the remaining two terms we get

$$b_n(K)=\frac{8K(-1)^i}{n^2\pi\alpha}\sin\frac{n\alpha}{4}\sin\frac{\alpha}{2}[\delta_{n/2,iK+1}+\delta_{n/2,iK-1}]$$
$$=\frac{4K^2(-1)^i}{n^2\pi^2}\sin\frac{n\pi}{2K}\sin\frac{\pi}{K}[\delta_{n/2,iK+1}+\delta_{n/2,iK-1}] \tag{A.10}$$

where we have used the definition $\alpha=2\pi/K$.

Now,

$$\sin\frac{n\pi}{2K}\delta_{n/2,iK\pm 1}=\pm(-1)^i\sin\frac{\pi}{K}\delta_{n/2,iK\pm 1}$$

and equation (A.10) reduces to

$$b_n(K)=\left(\frac{K}{\pi}\sin\frac{\pi}{K}\right)^2\left[\frac{\delta_{n/2,iK+1}}{(iK+1)^2}-\frac{\delta_{n/2,iK-1}}{(iK-1)^2}\right]. \tag{A.11}$$

Substituting this expression into equation (A.1) and expressing the sum over n as a sum over i we obtain equation ($1b$) which is equivalent to ($1a$).

Equation (3) is derived in a similar way, but the algebra is different.

The expansions for the integrals and derivatives, equations (6), (8)–(10), (12) and (13) are all straightforward. As we discussed in section IIB, the *second* derivatives, however, cannot be similarly expanded because they involve delta functions. Nevertheless we present here the "formal" differentiation alluded to at the end of that section because we first obtained equation (20) by this "non-kosher" method: replace the upper limits indicated in equations ($1b$) and ($3b$) by M and differentiate twice (there is no trouble for finite M)

$$D^2 \sin_K x = -\left(\frac{K}{\pi}\sin\frac{\pi}{K}\right)^2 \left\{\sin x + \sum_{i=1}^{M} [\sin(iK+1)x - \sin(iK-1)x]\right\}$$

$$= -\left(\frac{K}{\pi}\sin\frac{\pi}{K}\right)^2 \sin x \left\{1 + 2\sum_{i=1}^{M} \cos iKx\right\}. \tag{A.12}$$

Similarly,

$$D^2 \cos_K x = -\left(\frac{K}{\pi}\sin\frac{\pi}{K}\right)^2 \left\{\cos x + \sum_{i=1}^{M} [\cos(iK+1)x + \cos(iK-1)x]\right\}$$

$$= -\left(\frac{K}{\pi}\sin\frac{\pi}{K}\right)^2 \cos x \left\{1 + 2\sum_{i=1}^{M} \cos iKx\right\} \tag{A.13}$$

and the ratio of equation (A.12) to equation (A.13) gives

$$\frac{D^2 \sin_K x}{D^2 \cos_K x} = \tan x \tag{A.14}$$

which is equation (20). We have cancelled out the common factor $1 + 2\sum_{i=1}^{M} \cos(iKx)$ which gives

$$1 + 2\sum_{i=1}^{M} \cos(iKx) = 1 + 2\,\frac{\sin(MKx/2)\cos((M+1)Kx/2)}{\sin(Kx/2)}$$

$$= \frac{\sin[(M+\frac{1}{2})Kx]}{\sin(Kx/2)}. \tag{A.15}$$

The denominator in this expansion, $\sin(Kx/2)$, vanishes at each vertex $(x = (2\pi/K)\times(j-1), j = 1, 2, \ldots, K+1)$ which is precisely where equation (A.14) holds. The correct derivation and interpretation was given in section IIB.

Appendix B

To derive equation ($46a$) we use equation ($1a$):

$$\frac{1}{\pi}\int_0^{2\pi} \sin_K Lx \sin_L Kx \,\mathrm{d}x$$

$$= \left(\frac{K}{\pi}\sin\frac{\pi}{K}\right)^2 \left(\frac{L}{\pi}\sin\frac{\pi}{L}\right)^2 \sum_{i,j=-\infty}^{\infty} \left\{\frac{1}{(iK+1)^2(jL+1)^2} \times \frac{1}{\pi}\int_0^{2\pi} \sin(iK+1)Lx \sin(jL+1)Kx \,\mathrm{d}x\right\}. \tag{B.1}$$

The above integral vanishes unless $(iK+1)L$ equals $\pm(jL+1)K$. For $(iK+1)L=(jL+1)K$ we have

$$L=\frac{K}{1+(i-j)K} \tag{B.2a}$$

$$K=\frac{L}{1+(j-i)L}. \tag{B.2b}$$

Since K and L are positive integers greater or equal to one equation (B.2a) implies that $i-j\geqslant 0$ while equation (B.2b) implies that $j-i\geqslant 0$. Therefore $i=j$ and hence $K=L$.

For $(iK+1)L=-(jL+1)K$ we have

$$L=\frac{K}{pK-1} \tag{B.3a}$$

$$K=\frac{L}{pL-1} \tag{B.3b}$$

where $p\equiv-(i+j)$. While p can be any positive or negative integer depending on the values of i and j, only two values can be admitted for $L\geqslant 1$ and $K\geqslant 1$: $p=1$ and $p=2$, for which $K=L=2$ and $K=L=1$ respectively. Since $\sin_1 Lx=\sin_2 Lx=0$, the right-hand side of equation (B.1) vanishes for $K=L=1$ or 2.

For other values of K and L we have

$$\frac{1}{\pi}\int_0^{2\pi}\sin_K Lx\,\sin_L Kx\,\mathrm{d}x=\left(\frac{K}{\pi}\sin\frac{\pi}{K}\right)^4\sum_{i=-\infty}^{\infty}\frac{1}{(iK+1)^4}\,\delta_{K,L}. \tag{B.4}$$

Using the identity given in equation (11) the above equation reduces to equation (46a).

Equations (46b), (48a) and (48b) are derived in a similar way.

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